sums of yesteryear

A Compilation of Newspaper Columns Recalling the Old and Unusual In School Mathematics

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to accompany
"Mathematics for the Joy of It"
a presentation on mathematical recreations

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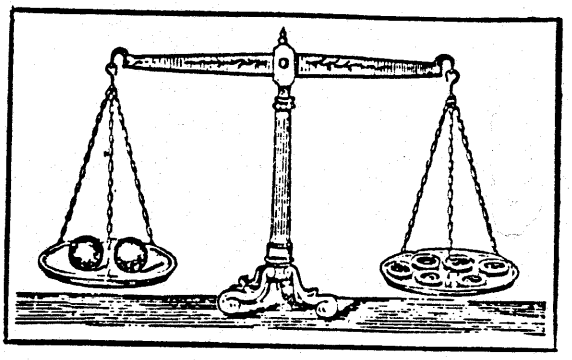
SUMS OF YESTERYEAR



FOREWORD

Old textbooks, especially old Arithmetics and Algebras, Victorian and older, are fascinating for the stories they tell as well as for the lessons they teach. Our first encounter with such works was in a vast, uncatalogued collection in a closed stack at our graduate school, Rutgers University. With photocopier--and, yes, with pencil and paper--we recorder the story problems with which early authors--self-styled"compilers"--motivated and "applied" their concepts. We shared these early problems in several periodicals, including The Rutgers University Library Journal, McGill Journal of Education, and Nova Scotia Journal of Education, but these periodicals are not with us in the North. What we are sharing here is two years of a newspaper column, "Sums of Yesteryear." The columns were written for classroom teachers, and appeared in The Teacher, official publication of the Nova Scotia Teachers Union. Our association with Nova Scotia Teachers College dates from 1969 through 1987, and these columns were a natural outgrowth of work in pre-service and in-service teacher education. When we had six hours in our study between hiring confirmation and flight departure (a not unusual initiation to the North, we now understand), the "Sums of Yesteryear" folder was one item which, on a whim, we tossed in the suitcase. We had an evening such as this in mind.

At this Mathematical Recreations evening arranged by the Iqaluit Reseach Centre, Science Institute of the Northwest Territories, we rather expect to "preach to the converted," and look forward to sharing (through this compilation) some of our favourite "old problems" with those most likely to enjoy them.

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1. The Encircling Thread

Textbook questions in solid geometry very often, though not always, can be viewed, and most readily attacked, from a two-dimensional perspective. A good illustration is found in Promiscuous Exercises in Mensuration in the compact but comprehensive **Theoretical and Practical Arithmetic** of schoolmaster Orlando Blanchard (Cazenovia, N.Y., 1854). The problem, as it turns out, calls for unusual application of the Pythagorean relation in an expression involving the square of pi. The challenge, essentially, is in spotting a "two-dimensional perspective" well suited to the attack.

"A cylinder being 3 feet in diameter and 20 feet high, how long is the thread wound spirally about it, so as to encircle it 10½ times, in ascending from the base to the top?"

Solution Notes

The lateral surface of a cylinder is equivalent to a rectangle. Or, in a most practical contest, the label from a tin can, removed, may be flattened to rectangular shape. Extending this observation to, essentially, a label wrapped a number of times, the following perspective on the Blanchard cylinder leads directly to solution.

The spiral thread rises 20 feet, the height of the cylinder, in extending horizontally 10½ times the circumference of the cylinder. The circumference of the circular is pi times the diameter, or 3 pi feet. Accordingly ("flattening the label"), the spiral length may be viewed as the hypotenuse of a right plane triangle having one side 20 feet and the other side 10½ times 3 pi, or 31½ pi feet. The Pythagorean relation then gives the hypotenuse, hence the spiral length, as 100.96+ feet.

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Editor's note:

Don Allen collects old textbooks, browses in library stacks, and finds particular fascination in historical roots of present-day school mathematics. His "Sums of Yesteryear" column begins a new series this issue. Don is on faculty at Nova Scotia Teachers College, Truro, and welcomes dialogue on old problems and methods of solution.

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2. A Mother's Share

Improbable old wills and such feature in the questions on proportion in many an early school Arithmetic. The challenge in such problems very often is in viewing them in proper perspective, recognizing and exploiting a best method of attack. One interesting representation of this problem type involves, essentially, a marriage contract. The question derives from Miscellaneous Examples in Benjamin Greenleaf, **National Arithmetic on the Inductive System** (Boston, 1867).

"A certain gentleman, at the time of his marriage, agreed to give his wife ⅔ of his estate, if, at the time of his death, he left only a daughter, and if he left only a son, she should have ⅓ of his property; but as it happened, he left a son and a daughter, in consequence of which the widow received in equity $2400 less than she would have received if there had been only a daughter. What would have been his wife's dowry if he had left only a son?"

Solution Notes

A daughter would receive one-half, and a son twice, the mother's share. Accordingly, with daughter and son, the daughter, mother, and son should share as ½:1:2, or 1:2:4. The mother thus receives 2/7 the estate. With only a daughter, however, the mother would have received 2/3 the estate. Hence 2/3 the estate less 2/7 the estate, or 8/21 the estate, is $2400; and the estate is $6300. Accordingly, with only a son, the mother would receive 7 1/3 of $6300, or $2100, Greenleaf's answer of 1867.

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3. The Tub Will Be Full

Cistern problems were a favourite form of "rate problem" in many an early Arithmetic. Such questions frequently were quite ingenious, and normally required careful, involved fraction manipulation. The following cistern problem derives from Charles Davies, **Practical Arithmetic** (New York, 1869), a comprehensive treatise. Featured is a "water-tub" into which water flows from a pipe and from which water discharges from a tap.

"A water-tub holds 147 gallons; the pipe usually brings in 14 gallons in 9 minutes; the tap discharges at a medium, 40 gallons in 31 minutes. Now, supposing these to be left open, and the water to be turned on at 2 o'clock in the morning; a servant at 5 shuts the tap, and is solicitous to know at what time the tub will be filled, in case the water continues to flow."

Solution Notes

Through the **pipe**, water **enters** the tub at 1 5/9 gallons per minute, while through the **tap**, while it is open, water flows **out** at 1 9/31 gallons per minute. Accordingly, with both pipe and tap in operation, water in the tub increases at a rate of 1 5/9 - 1 9/31, or 74/279 gallon per minute. Pipe and tap flow from 2 o'clock to 5 o'clock, an interval of 180 minutes, putting into the tub a net amount of 180 x 74/279, or 47 23/31 gallons. The tub having 147 gallon capacity, this leaves room for 147 - 47 23/31, or 99 8/31 gallons. The tap shut, the tub then continues to fill at 1 5/9 gallons per minute, 99 8/31 gallons being added in 99 8/31 divided by 1 5/9, or 63 351/434 minutes. The tub, accordingly, will be full at 6:03 351/434 (or 6:03:48 114/217, Davies' wondrously precise answer of 1869).

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4. Order on the Poll

Apart from problems deriving from cumbersome manipulation of old and non-decimal monies and measurements — in which early textbooks abound — "real-world" exercises in fractional parts must always be a challenge to the textbook author. From Cambridge scholar Hamblin Smith, in the Examination Papers compendium of his **Treatise on Arithmetic**, this challenge elicited a wholly worthy response. The Canadian adaptation by Thomas Kirkland and William Scott (Toronto, 1890) includes this fine problem on multiple voting patterns with regard to five electoral candidates in a well-contested constituency.

"In a constituency, in which each elector may vote for two candidates, half of the constituency vote for **A**, but divide their votes among **B, C, D, E**, in the proportions of 4, 3, 2, 1; of the remainder, half vote for **B**, and divide their votes among **C, D, E**, in the proportions of 3, 1, 1; two-thirds of the remainder vote for **D** and **E**, and 540 do not vote al all; find the order on the poll, and the whole number of electors.

Solution Notes

An algebraic solution has the advantage of relative brevity. Let the number of electors be n.

A receives, in all, n/2 votes.

From these same electors, **B, C, D, E**, receive, respectively, 2/5 x n/2, 3/10 x n/2, 1/5 x n/2, 1/10 x n/2 votes.

B also receives, from the remaining electors, 1/2 x n/2, or n/4 votes.

Thus, **B** receives, in all, n/5 + n/4, or 9n/20 votes.

From this second group of electors, **C, D, E**, receive, respectively, 3/5 x n/4, 1/5 x n/4, 1/5 x n/4 votes.

Thus, **C** receives, in all, 3n/20 + 3n/20, or 3n/10 votes.

From this third group of electors, **D, E**, each receives 2/3 x n/4, or n/6 votes.

Thus, **D** receives, in all, n/10 + n/20 + n/6, or 19n/60 votes.

Correspondingly, **E** receives, in all, n/20 + n/20 + n/6, or 4n/15 votes.

The five candidates receive, in all, 1 5/6 n votes.

The remaining n/6 votes represent 540 x 2, or 1080 uncast votes.

Accordingly, n = 6480, the number of electors, and the order on the poll is **A** (3240 votes), **B** (2916 votes), **D** (2052 votes), **C** (1944 votes), **E** (1728 votes).

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5. A Walkingame Sampler

The Tutor's Assistant, Francis Walkingame's "Compendium of Arithmetic," had known decades of good service before our battered copy of the 1846 London "new edition" found its way to a Nova Scotia community later that decade. Offered in its day as "A Complete Question Book," **Crosby's Walkingame** (as the revision was known) covered the widest possible range of arithmetic topics. The problems, however, were not always of the sophistication frequently encountered in later compilations. The following are perhaps representative of the more interesting questions. Little difficulty will be encountered, but careful reading is advised.

1. A snail, in getting up a May-pole, only 20 feet high was observed to climb 8 feet every day, but every night it came down again 4 feet; in what time, by this method, did it reach the top of the pole?

2. The spectator's club of fat people, though it consisted of no more than 15 members, is said to have weighed no less than 3 tons; how much, at an equality, was that per man?

3. An ancient lady being asked how old she was, to avoid a direct answer, said, I have 9 children, and there are 3 years between the birth of each of them; the eldest was born when I was 19 years old, which is now exactly the age of the youngest; how old was the lady?

4. A sheepfold was robbed three nights successively; the first night half of the sheep were stolen, and half a sheep more; the second night half the remainder were lost, and half a sheep more; the last night they took half that were left, and half a sheep more; by which time they were reduced to 20; how many were there at first.?

Solution Notes

1. The classic problem of the climbing, backsliding snail survives principally in collections of puzzles and trick questions. Essentially, the snail climbs 8 feet the first day, then nets 4 additional feet each succeeding day, reaching the top of the pole in 4 days.

2. Fifteen club members "weigh in" at 3 tons, or 60 hundredweights. Each averages, therefore, "4 cwts.," which is Walkingame's answer. These, however, undoubtedly would be long tons (2240 pounds) and long hundredweights. Four long hundredweights, each of 8 stones of 14 pounds, bring the members' average weight to a hefty 448 pounds.

3. Adding 19 initial years, eight 3-year intervals between births, and 19 further years, gives the lady's current age, a hardly "ancient" 62.

4. Algebra here offers useful shorthand. Begin with n sheep. The first night, $n/2 + 1/2$ sheep are stolen, so $n - (n/2 + 1/2)$, or $n/2 - 1/2$ remain. The second night, $1/2 (n/2 - 1/2) + 1/2$, or $n/4 + 1/4$ are stolen, so $(n/2 - 1/2) - (n/4 + 1/4)$, or $n/4 - 3/4$ remain. The third night, $1/2 (n/4 - 3/4) + 1/2$, or $n/8 + 1/8$ are stolen, so $(n/4 - 3/4) - (n/8 + 1/8)$, or $n/8 - 7/8$ remain. These $n/8 - 7/8$ sheep being the 20 sheep referred to, $n = 167$, Walkingame's answer.

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6: Around the Island

John Gough's **Treatise of Arithmetic**, in a rare North American edition (Philadelphia, 1788), purported to contain "Every Thing Important to the Study of Abstract and Applicate Numbers." The work is a rich source of early "word problems." Included is one problem type, here considered, designed to challenge student comprehension of rates, distances, and multiples.

"There is an island which is 36 miles in compass: Now, if at the same time, and from the same place, 2 footmen, A and B, set forward to travel around said island, and follow one another in such a manner, that A travels every day 9 miles, and B 7 miles: The question is to know in what space of time they will meet again, and also how many miles and also how many times around the island each footman will have travelled?

Solution Notes

At stipulated rates of travel, A completes, each day, 1/4 circuit and B completes 7/36 circuit. After n days, A has completed 1/4 n, and B has completed 7/36 n circuits. When next they meet, B, the slower traveller, will have completed one circuit less than A. In terms of numbers of completed circuits, $1/4\ n - 7/36\ n = 1$. Accordingly, $n = 18$. They meet after 18 days, at which point A has travelled 162 miles, or 4½ times around the island, and B has travelled 126 miles, or 3½ times around the island.

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7. The Wagering Pedestrian

A delightfully contrived "rate" problem in Kelland's **Algebra** (Edinburgh, 1869) has built-in potential for a succession of elementary algebraic surprises — agreeable surprises at that! Deriving from a relatively early (1823) English college test item, the unlikely nineteenth-century tale of the wagering, backwards-walking "pedestrian" proceeds to surprise its would-be solver by successively (1) reducing to a single variable, (2) having quadratic terms "vanish" (as engineers say), and (3) yielding an improbably simple numerical solution. A relatively straightforward exercise in the tidying up of equated rational expressions, Kelland's "wagering pedestrian" problem more than amply rewards an initial moment of incisive thinking and a necessary minimum of careful and correct manipulation.

One slip, of course, and those "agreeable surprises" very likely fail to follow!

A pedestrian, finding that he could walk forwards four times as fast as he could backwards, undertook to walk a certain distance (one-fourth of it backwards) in a given time. But the ground being bad, he found that his rate per hour backwards was one-fifth of a mile less than he supposed, and that to have won his wager he must have walked forwards two miles an hour faster than he did. What is his rate per hour backwards?

Solution Notes

Suppose, to minimize fractions, that the pedestrian could walk backwards at r miles per hour. His corresponding rate forwards would be 4 r miles per hour. Denote by 4d the number of miles in the "certain distance." The pedestrian undertook to walk backwards d miles and forward the remaining 3d miles.

Now, "the ground being bad," the pedestrian's backwards rate became, we are told, r - 1/5 miles per hour. His corresponding forwards rate necessarily would be 4r + 2 miles per hour.

Each walking time being a quotient of distance walked divided by rate, the pedestrian had undertaken to walk the "certain distance" (the 4d miles) is a number of hours equal to $3d/4r + d/r$. "The ground being bad," he required $3d/(4r + 2) + d/(r - 1/5)$ hours. Equating equal numbers of hours, $3d/4r + d/r = 3d/(4r + 2) + d/(r - 1/5)$. Dividing through by, and wholly eliminating, the non-zero d, $3/4r + 1/r = 3/(4r + 2) + 1/(r - 1/5)$. Simplifying, $12r(5r - 1) + 20r(4r + 2) = 7(20r^2 + 6r - 2)$. Second-degree terms cancelling, r = 1, uniquely! The pedestrian's rate backwards was exactly one mile per hour; forwards, four miles per hour. On "bad ground" he covered, backwards, four-fifths mile per hour; forwards, to win his wager, six miles per hour.

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9. A Six-day Week

A long work day and a six-day week were the norm in the age from which many interesting labour problems derive. Accordingly, an "examination-paper" question posed by Bishop Colenso in his **Arithmetic** (London, 1870) takes for granted that "a certain manufactory" worked its labour force a six-day week. As often might happen in a mid-Victorian school Arithmetic, consideration of an unwieldy fraction (in this instance an improbable "23/83 of an hour") quickly compounds itself to four-digit numerators and denominators. In Colenso's example, however, virtually eveything "cancels," the final answser is disconcertingly simple, and the problem (one might say) has a happy ending.

"In a certain manufactory, 158 men execute a certain piece of work in a week; but if the abilities of 2 of them had been, respectively, 1/7 and 1/9 less than ordinary, and the abilities of 2 others 3/5 and 3/8 more, the work could have been finished 23/83 of an hour sooner. How many hours a day did the men work?

Solution Notes

We compute the length of the work week, assume six working days, and obtain Colenso's answer.

Suppose 158 "men of ordinary ability" to have worked h hours a week, doing 158h "ordinary man hours" of work.

The alternative under consideration is that 154 men of ordinary ability work (h - 23/83) hours, doing 154(h - 23/83) ordinary man hours of work, and that one man of 6/7 ordinary ability, one of 8/9 ordinary ability, one of 1 3/5 ordinary ability, and one of 1 3/8 ordinary ability, also work (h - 23/83) hours, doing (6/7 + 8/9 + 1 3/5 + 1 3/8) (h - 23/83), or 4 1817/2520 (h - 23/83) ordinary man hours of work.

Equating work done:

158h = 154(h - 23/83) + 4 1817/2520 (h - 23/83.)

Accordingly, h = (23 × 399 977)/(83 × 1817), or 61 hours, exactly.

The men worked 61 hours; that is, six days of 10 1/6 hours, exactly.

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10. A Globe Filled with Wine

A contrived sort of mensuration question which equates numbers which ordinarily never would be compared was popular in some earlier textbooks. Construction of such a question could call for considerable ingenuity, one must concede. In his **Practical Application of the Principles of Geometry to the Mensuration of Superficies and Solids** (New Haven, 1811), Jeremiah Day (later president of Yale) provides an instructive example which solves in unusual fashion. Day's question equates spherical area and volume considerations, but computation is disconcertingly straightforward. Irrationals vanish, higher powers cancel, and a linear relation leads to an exact decimal result.

The question involves the traditional United States wine gallon of 231 cubic inches.

"If a metallic globe is filled with wine, which costs as much at 5 dollars a gallon, as the globe itself at 20 cents for every square inch of its surface; what is the diameter of the globe?"

Solution Notes

Suppose the globe to be d inches in diameter. The surface area of the globe is pi times d^2 square inches, and the cost of the globe is 20 pi d^2 cents.

The capacity of the hollow globe is its volume, one-sixth pi d^3 cubic inches. At 231 cubic inches to the wine gallon, the globe will hold 1/1386 pi d^3 gallons. At 5 dollars a gallon, this wine would cost 250/693 pi d^3 cents.

Equating numbers of cents, pi factors cancel, as do factors of d^2. We are left with the linear relation 250/693 d = 20. Accordingly, d, the number of inches in the required diameter, is 55.44, exactly.

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11. Sharing the Butterball

Of the quaint category of early mensuration problems involving joint ownership, and therefore eventual division, of an improbable object of a relevant mathematical shape, Sangster's "examination problem" of the five-inch butterball surely must be unique. To be found in the 1880 New Brunswick edition of his popular secondary-school work, John Herbert Sangster, M.A., M.D., **National Arithmetic** (St. Stephen, N.B. preface dated 1859), the butterball problem, as posed, calls for four equal shares, taken "separately from the surface of the ball." Care and observation in manipulation of rational and irrational expressions should reduce traditional hand-computation considerably.

"Four women own a ball of butter, 5 inches in diameter. It is agreed that each shall take her share separately from the surface of the ball. How many inches of its diameter shall each take?

Solution Notes

The volume of a sphere being 4/3 pi times the cube of the radius or 1/6 pi times the cube of the diameter, the initial volume of the butterball is 125/6 pi, exactly, or 65.4498 + cubic inches. Volumes of concentric inner spheres corresponding to one-, two-, and three-fourths the butter will be 1/4(125/6 pi), 2/4(125/6pi), and 3/4(125/6pi); that is, 16.3624+, 32.7249+, and 49.0873+ cubic inches. Corresponding cubes of radii simplify to 125/32, 250/32, and 375/32 cubic inches. Corresponding diameters, accordingly, are 5/2 the cube roots of 2, 4, and 6, exactly; or 3.1498+, 3.9685+, and 4.5428+ inches.

The four women, therefore, proceed as follows. The first woman reduces the 5 inch ball to 4.5428+ inches, taking one-fourth the butter and removing 0.4571 + inches from the ball's diameter. The second woman then reduces the 4.5428 + inch ball to 3.9685 + inches, taking one-fourth the original butter and removing 0.5742+ inches from the ball's diameter. The third woman then reduces the 3.9685+ inch ball to 3.1498+ inches, taking one-fourth the original butter and removing 0.8187 inches from the ball's diameter. The fourth woman then claims the remaining one-fourth of the butter, a reduced ball 3.1498+ inches in diameter.

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12. The Bishop's Oxen

A distinctive type of rate problem found in a number of early Arithmetics involves livestock being set to graze in a meadow for extended periods of time. Assumed is that as grazing continues, grass grows back, at a uniform rate. The livestock invariably are oxen.

With a full set of exercises and several "model solutions," an 1870 London **Arithmetic** presents the "growing grass" problem type at what undoubtedly was its fullest development. "Questions on the uniform consumption of uniformly growing produce" are as compiled by The Right Reverend J.W. Colenso, D.D., Lord Bishop of Natal. Our choice is a question of perhaps average difficulty, having an integral solution. Fraction work is somewhat unwieldy, as is usual with this problem type, but arithmetic methods do suffice.

"If 29 oxen would eat up a field of grass in 7 weeks, or 25 oxen would eat up the same field in 9 weeks, — the grass growing uniformly, how many oxen would do it in 6 weeks?"

Solution Notes

The grass initially on the field plus 9 weekly growths would be eaten by 25 oxen in 9 weeks, or by 1 ox in 9x25, or 225 weeks. (One ox in one week, therefore, eats 1/225 grass plus 1/25 growth.)

The grass initially on the field plus 7 weekly growths would be eaten by 29 oxen in 7 weeks, or by 1 ox in 7x29, or 203 weeks. (One ox in one week, therefore, eats 1/203 grass plus 1/29 growth.)

The difference between 9 growths and 7 growths, or 2 growths, would be eaten by 1 ox in 225-203, or 22 weeks. Accordingly, 1 ox in 1 week eats the equivalent of 1/11 growth.

Further, equating equivalent expressions, 1/225 grass + 1/25 growth = 1/203 grass + 1/29 growth. On simplification, the grass initially on the field equals 11 5/11 weekly growths.

That grass plus six weekly growths is to be eaten in 6 weeks. Equivalently, 17 5/11 growths are to be eaten in 6 weeks, or 2 10/11 growths in 1 week. Each ox eats 1/11 growth in 1 week. This much eating will necessitate 2 10/11 divided by 1/11, or 32 oxen.

Further Considerations

The author's business card for some years has featured an earlier but similar problem, one which able secondary students have found hard. Deriving from Frederick Emerson, **North American Arithmetic** (Boston, 1839), the question resembles the Lord Bishop's, but Emerson's fraction work is formidable and the solution non-integral.

"If 12 oxen eat up 3½ acres of grass in 4 weeks, and 21 oxen eat up 10 acres in 9 weeks, how many oxen will eat up 24 acres in 18 weeks; the grass being at first equal on every acre, and growing uniformly?"

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13. Fibonacci of Canandaigua

A pair of newborn rabbits will mature in one month. The second month that pair will produce a further pair. Given one pair of newborn rabbits, after n months, according to this simplified rendering of rabbit facts of life, how many pairs of rabbits, in all, will there be.

Fibonacci (Leonardo of Pisa), the greatest mathematician of his age, pondered this unusual question early in the thirteenth century. The sequence of positive integers to which the question gives rise (1, 1, 2, 3, 5, 8, 13, ...), now universally known as the Fibonacci sequence, is rich in natural and theoretical applications and interpretations.

Fibonacci's rabbit problem and associated "Fibonacci numbers," accordingly, receive attention in many elementary textbooks, as enrichment or discovery mathematics, as a lead-in to difference equations and recursion relations, or as an exercise in mathematical induction.

What no modern textbook mentions, very possibly because no one remembers, is the "multiplying cows" variant of the Fibonacci rabbits, as popularized by the American, Ostrander, 160 years ago!

Tobias Ostrander, versatile and imaginative "teacher of mathematics," published his **Elements of Number, or Easy Instructor** in Canandaigua, New York State, in 1823. The compact little work proves a remarkable compilation and storehouse of contemporary, frequently nonroutine, word problems. That Ostrander had read Fibonacci and as a consequence had devised the cattle problem cannot be confirmed. Versions of the cattle problem being found in later printings of major English collections, a common, probably English, "ancestry" to the question would seem more likely. Featuring it in his back-of-the-book Miscellaneous Questions, Ostrander may, nonetheless, have introduced or popularized the cows and calves problem in America.

The following is Ostrander's version of 1823.

"How many head of cattle may come from a heifer calf in 20 years, provided they are all of the feminine gender, and each one have a calf when she becomes 3 years old, and after that have a calf every year?"

Include in your total "the old cow."

The following is the wording with which the problem appears in Bonnycastle's **Scholar's Guide to Arithmetic** (London, 1843). Virtually identical wording is found in Taylor's **Arithmetician's Guide** (London, 1853).

"Suppose a farmer has a calf, which at the end of three years begins to breed, and afterwards brings a female calf every year; and that each calf begins to breed in like manner at the end of three years, bringing forth a cow-calf every year; and that these last breed in the same manner, etc.; it is required to determine the owner's whole stock at the end of 20 years?"

Solution Notes

Draw up a table. Show, by years, numbers of calves, one-year-olds, two-year-olds, and mature (reproducing) cows. Elementary but significant insights should derive from tabulated number patterns. For example, (i) the number of mature cows in year n equals the total number of cattle in your n-3; (ii) the increase each year is the number of newborn calves; (iii) the number of newborns, and hence the increase, is the number of mature cows giving birth.

Careful tabulation leads to Ostrander's twenty-year figure, "1278, including the old cow." This answer breaks down to 189 two-year-olds, 277 one-year-olds, 406 mature cows, and 406 newborn calves.

Number of cattle, clearly, is a function of number of years. Denote by f(n) the number of cattle after year n. The "Ostrander sequence" (1, 1, 1, 2, 3, 4, 6, 9, 13, ...) satisfies the difference equation $f(n + 3) = f(n + 2) + f(n)$, subject to initial conditions $f(1) = f(2) = f(3) = 1$. The twentieth term of the associated sequence is indeed 1278.

Like Fibonacci's rabbits, Ostrander's cows and calves permit, even invite, "variations on a theme," but old Arithmetics appear to offer no other such problems.

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14. A Study in Frailty

The hand-held calculator, the marvel of our generation, can make short work of horrendously involved arithmetic computation which the Victorian textbook author may have felt worthwhile in itself. No calculator lessens the

need for careful reading, criticial thinking, and a sound plan of attack, in arithmetic problem solving, however. Here is a prime illustration. From Greenleaf's **Practical Arithmetic** (Boston, 1867), this delightful study in human frailty was intended, any moralizing aside, for pencil-and-paper solution, many pages of it. Only the rare student of "practical arithmetic" would have had access to the seven (or more) place logarithms that would immeasurably have shortened the work.

Allow yourself the hand-held calculator, by all means, but at least give thought to the labour that it saves. Seven decimal places are required in solution.

"A servant draws off a gallon on each day, for 20 days, from a cask containing 10 gallons of wine, each time supplying the deficiency by the addition of a gallon of water and then, to escape detection, he again draws off 20 gallons, supplying the deficiency each time by a gallon of wine. How much water still remains in the cask?"

Solution Notes

Consider first the "drawing off" of wine. Each "drawing off" of one of the ten gallons in the cask, that is one-tenth, and replacement with water, removes one-tenth and leaves nine-tenths of the wine then present. Twenty successive such "drawing off" leave wine in the theoretical amount of ten gallons times the twentieth power of nine-tenths; that is, 1.215 765 5+ gallons. The remaining 8.784 233 4+ gallons in the cask are water. Each subsequent "drawing off" of one of the ten gallons in the cask, that is one-tenth, and replacement now with wine, removes one-tenth and leaves nine-tenths of the water then present. Twenty successive such "drawings off," accordingly, leave water in the amount of 8.784 233 4+ gallons times the twentieth power of nine-tenths; that is, 1.067 957 7+ gallons.

To eight digits, few of them "significant," this agrees with Greenleaf's, "1.0679577 gallons, or more than a gallon and half a pint."

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15. Dispatches for the Army

"Never judge a textbook by its title" was as sound a maxim when James Thomson penned his highly successful **Practical Arithmetic** (Cincinnati, 1848) as (sad to relate) it is today. However, to his credit, in "uniting the inductive with the synthetic mode of instruction," Thomson did live up to his promise to stress the practical, as the work's "miscellaneous examples" particularly attest.

Practical and in touch with rapidly changing times were some of the more interesting of the many such "story problems." Consider this rather unusual rate problem involving military dispatches to the Mexican war zone.

"A messenger traveling 8 miles an hour, was sent to Mexico with dispatches for the army; after he had gone 51 miles, another was sent with countermanding orders, who could go 19 miles as quick as the former could go 16: how long will it take the latter to overtake the former; and how far must he travel?"

An algebraic solution should follow readily, but any additional effort needed for a purely arithmetic approach may prove worthwhile.

Solution Notes

The problem, instructively, may be viewed as essentially one of relative speeds.

The first messenger, we are told, travels 8 miles an hour.

The second messenger covers 19 miles while the first covers 16; that is, in 2 hours. The second messenger, therefore, travels 9½ miles an hour.

Accordingly, from the time he starts out, the second messenger is overtaking the first at 9½ - 8, or 1½ miles an hour.

The second messenger must overtake for a distance of 51 miles. Gaining 1½ miles each hour, he will accomplish this in 51 divided by 1½, or 34 hours.

The headstart of the first mesenger, 51 miles, at his speed of 8 miles an hour, represents additional travel time of 51 divided by 8, or 6⅜ hours.

Accordingly, in 34 hours, the second messenger, at 9½ miles an hour, covers 323 miles. In 34 + 6⅜, or 40⅜ hours, the first messenger, too, covers 323 miles, which serves to check our result.

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16. Coat and Pantaloons

Problems which (to us) lend themselves to algebraic solution in one or more variables, are encountered not infrequently in early Arithmetics. When such is their origin, however, there can be little doubt that an arithmetic solution was what the textbook author had in mind.

A good illustration is found in Charles Davies, **Intellectual Arithmetic** (New York, 1858), a work which approached "the science of numbers" with (according to the subtitle) "especial reference to mental training and development." The problem in question presents two independent relationships between the prices of items of male wearing apparel. The reader may value the "mental training" inherent in restricting his solution and verification to a strictly arithmetic approach.

"A man's coat cost him 1½ times as much as his pantaloons, his pantaloons cost 6 dollars less than his coat: what was the cost of the pantaloons? What was the cost of the coat?"

Solution Notes

Since the coat cost 1½ times as much as the pantaloons, we may say that the difference between the cost of the coat and the cost of the pantaloons is one-half the cost of the pantaloons. This difference, we are told, is 6 dollars. Accordingly, the pantaloons cost twice 6, or 12 dollars. The coat cost 1½ times 12, or 18 dollars. The difference between 18 and 12 dollars is 6 dollars, which serves to verify both results.

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17. Gunshots at the Station

Improbable precision and unattainable "exactness" characterized the questions and the answers of most early school Arithmetics. "Close" never was good enough! An answer that deviated, however slightly, when compared with "the back of the book" was suspect, to say the least.

A rather distinctive "rate" problem — with an exact answer — should well illustrate the point. From Colenso's **Arithmetic** (London, 1870), it provides a numerical answer to apparent hundredth of a second precision, but gives no hint as to how the answer is obtained.

The reader, accordingly, may enjoy Colenso's "examination-paper" challenge — knowing that "rounding" would have had no place in solution and that difference of opinion, however slight, would arise from misinterpretation of the problem, or else miscalculation!

"The interval between the firing of two guns, at a railway station, was 6 minutes, and a passenger in a train, approaching the station at a uniform rate, heard the second report 5 min. 51 sec. after hearing the first. Now, suppose the sound of the train's approach to have been audible at the station when the train was 2 miles off, how soon after that did the train pass the station, — sound travelling 1125 feet per second?"

Colenso's answer? — 6 min. 6.08 sec.

Solution Notes

The shots occurred 6 min. apart, but were heard on the train 5 min. 51 sec. apart, we are told. Accordingly, in 5 min. 51 sec. (351 sec.) the train must have covered the distance that sound travels in 9 sec., namely 9 x 1125, or 10 125 feet. Also, at 9/351, or 1/39 the speed of sound, the train was moving 1125/39, or 28 11/13 feet per second (19 191/286 miles per hour). Sound made by the train was audible when the train was 2 miles, or 10 560 feet, from the station. This distance, at this rate, would be covered in 10 560 divided by 28 11/13, or 366 2/25 seconds. (the sound of the train originated 10, 560/1125, or 9 29/75 seconds earlier, of course, but this fact is not relevant to the solution.) Accordingly, the train will pass the station in 6 min. 6 2/25 sec., equivalent to Colenso's exact decimal solution.

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18. A "Rate Problem" Trio

Rate problems of several elementary types served to apply and extend "fraction" and "ratio" concepts in the typical Victorian school Arithmetic. By turn of century, however, such questions were featured less prominently, and manipulation of awkward rationals had been conspicuously reduced. "Miscellaneous Problems" in the 1895 Kirland and Scott **Elementary Arithmetic on the Unitary System** (Toronto, Gage Mathematical Series) included significant groupings of several such problem "types," however. The questions were briefly stated and relatively straight forward, but each did call for a moment of initial thought. Recalling a still earlier decade in North American school mathematics, the three "rate" problems which follow are perhaps representative. Each yields to an elementary algebraic approach — or, with a certain elegance, to a Kirkland and Scott "unitary system" method of attack.

Rowing with and against the current provides the basis for the first such problem type.

1. A can row 4½ miles down stream in an hour. Without the aid of the stream it would have taken him an hour and a half. How long would it take him to return?

A second problem type involves a round-trip journey made at differing, unrelated rates.

2. How far may a person ride in a carriage going at the rate of 8 miles an hour, provided he is gone 11 hours and walks back at the rate of 3 miles an hour?

Relative positions of hour and minute hands on a clock face provided the basis for a rate problem "type," exemplified by the following question.

3. It is between four and five, and the hands of a watch are equally distant from the figure 4; what is the first time? the second time?

Solution Notes

1. Requiring 1½ hours to row 4½ miles in still, A rows at 3 miles per hour. Downstream, the stream's rate adding to his own, A covers 4½ miles in an hour. The rate of the stream, therefore, is 1½ miles per hour. Upstream, his rate in still water diminished by the stream's rate, A covers 3 - 1½, or 1½ miles, each hour. Accordingly, to return 4½ miles upstream will require 4½ divided by 1½, or 3 hours.

2. Each mile travelled on the journey will take 1/8 hour in the carriage, plus 1/3 hour, subsequently, returning on foot, a total of 11/24 hour. **Each** hour (the "unitary" approach) the persons covers 24/11, or 2 2/11 miles. Eleven hours, therefore, permit a ride and return of 11 times 2 2/11, or 24 miles.

3. Algebra, here, is brief and effective. Suppose hour and minute hand are equally distant from "4" at m minutes after the hour. The minute hand of a watch travels m minute divisions in m minutes. The hour hand travels m/12 minute divisions in the same time. The hands first are equally distant from "4" when the hour hand is m/12 divisions past "4" and the minute hand 20-m divisions before "4". Equating, m/12 = 20 - m, and m = 18 6/13. The hands next are equally distant from "4" when the hour hand is m/12 divisions past "4" and the minute hand m - 20 divisions past "4." Equating, m/12 = m - 20, and m = 21 9/11. The two times are 4:18 6/13 and 4:21 9/11.

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19. The Value of Money

Old Arithmetics can be eloquent on the value of money. Their story problems stressed useful applications, very often to the world of commerce. Questions involving buying and bartering, accordingly, would document

prevailng price levels, and profits to be made in trade. "Promiscuous Questions" in Charles Davies, **Intellectual Arithmetic** (New York, 1858), include an attractive selection of trade-related problems. Several of these questions, though elementary, provide insight into what money would buy, and how money was earned, back "when a dollar was a dollar." The following four "story problems" may be considered representative.

1. A farmer buys a pig, for which he pays 3 dollars, and also a sheep and a cow; the cow cost 3 times as much as the pig and the sheep, and the sheep cost 5 times as much as the pig: what was the cost of each?

2. A man bought a horse, saddle, and bridle: he paid 2 times as much for the saddle as for the bridle, and 11 times as much for the horse as for saddle and bridle both; in all he paid $108: how much did he pay for each?

3. A merchant sold 6¾ yards of cloth at 4 dollars a yard, and took his pay in equal quantities of rye and wheat, the former at 50 cents, and the latter at $1 a bushel: how much wheat did he receive?

4. A market woman bought a certain number of eggs at the rate of 4 for 3 cents, and sold them at the rate of 5 for 4 cents, by which she made 4 cents. What did she pay apiece for the eggs? What did she make on each egg sold? How many did she sell to make 4 cents?

Solution Notes

Solution in each instance depends primarily on organizing the relevant facts.

1. The pig cost $3. The sheep cost five times as much as the pig, or $15. The cow cost three times as much as the pig and the sheep, or 3 times $18, or $54.

2. The saddle was equivalent in price to two bridles. The horse was equivalent to 11 saddles and 11 bridles; that is, to 33 bridles. Horse, saddle, and bridle were equivalent to 1 + 2 +33, or 36 bridles, and cost $108. The bridle, accordingly, cost $108 divided by 36, or $3, the saddle cost $6, and the horse cost $99.

3. The cloth, 6¾ yards at $4 a yard, was worth $27. Trading for equal quantities of rye at 50¢ a bushel and wheat at $1 a bushel would necessitate that 2/3 of $27, or $18, be for wheat. At $1 a bushel, the merchant would receive 18 bushels of wheat.

4. Each egg was bought at 3/4¢ and sold at 4/5¢, yielding a profit of 1/20¢. To make four cents the market woman sold 4 divided by 1/20, or eighty eggs.

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20. Impressive Sums

Particular fascination with the way in which numbers could accumulate and totals "grow" is evident in "story problems" of many early schoolbooks. Terms and sums of "progressions" (arithmetic and geometric) were treated in most early Arithmetics as well as Algebras, with the usual formulas presented or developed. A selection of work problems from such pioneer texts shows that even arithmetic sequences could, on occasion, give rise to impressive final terms and hence sums. Solution of such problems no doubt was primarily "mechanical — calling, in the main, for identification and substitution in the right formula. The questions and answers have a certain appeal, nonetheless, not infrequently offering unconscious insights into matters considered of interest to the life and times.

"Charity" very likely had different connotation in the period prior to the coining of "welfare state." With just a hint of moralizing, Yale president Jerimiah Day raised the following question in his **Introduction to Algebra** (New Haven and New York, 1839).

1. A person put into a charity box a cent the first day of the year, two cents the second day, three cents the third day, etc. to the end of the year. What was the whole sum for 365 days?

Wages are the subject of a similar "progression" question in E. E. White, **Complete Arithmetic** (Cincinnati and New York, 1870).

2. A man agreed to dig a trench 50 yards long for 2 cents for the first yard, 5 cents for the second yard, 8 cents for the third, and so on, the price for each yard being 3 cents more than that of the preceding yard: what did he receive

for digging the last yard? For digging the trench?

An early and revealing example of this problem type will be found in Bejamin Workman, **The American Accountant, or Schoolmaster's New Assistant** (Philadelphia, 1789), the subject matter alluding, one suspects, to eighteenth-century reality.

3. Supposing a press gang having a warrant to press for 30 days, press the first day 300 men, and every succeeding day 10 more than the former, how many will they raise in the 30 days?

An attractively simple application of the arithmetic progression sum is included by George Hull in his **Complete Arithmetic** (Philadelphia, 1895).

4. How many times does a common clock strike in a week?

Solution Notes

Such questions call for little more than formula application, but some do invite pertinent observations.

1. The last day's contribution being 365 cents, the number of cents in the required total is sigma 365, the sum of the first 365 positive integers. As an arithmetic progression sum, this is the number of terms (days, 365) times the mean of the first and last terms (1 + 365)/2, or 183. That is, 66795 cents, or $667.95.

2. Numbers of cents earned form an arithemetic progression, first term = 2, common difference = 3. The 50th term is the 1st plus 49 times the common difference; that is, 149. The sum is given by the number of terms times the mean of the first and last, 50 x 51/2, or 3775. The man received $1.49 for digging the last yard and $37.75 for digging the trench.

3. Pressing 300 the first day, 10 more each succeeding day, and (accordingly) 590 on the 30th day, the gang "recruits" in all 30 x 890/2, or 13 350 men.

4. Fourteen "cycles" of 1 + 2 + ... + 12 (sigma 12), or 78 strikes, gives 1092 strikes in 7 days.

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21. Staggering Numerical Heights

A nineteenth-century textbook preoccupation with rapidly mounting sums is particularly evident in the rather remarkable class of "story problems" designed to illustrate and apply geometric progression summation. Pupils soon found that deceptively "simple" problem situations of this type could force pencil-and-paper calculations to staggering heights. The practicality of most such questions may be doubtful at best, and solution primarily involved formula recognition and computation. Numerial results could be quite surprising, however, and in that sense at least, such questions were instructive. Three simple but interesting illustrations of the problem type follow.

White, in his **Complete Arithmetic** (1870), poses this question about a son's increasing worth:

1. A father gave his son 50 cents on his 12th birthday, and agreed to double the amount on each succeeding birthday to and including the 21st: how much did the son receive on his 21st birthday? How much in all?

The fiscal fate of one "gentleman, ignorant in numbers" is described by Ostrander in his **Elements of Numbers** (1823) as follows:

2. A gentleman, ignorant in numbers, wishing to purchase 15 elegant horses, agreed with a sharper to furnish him with the number on the following conditons: that he should pay him 4 cents for the first, 16 for the second, 64 for the third, &c., in quadruple geometrical proportion; I demand the amount of the 15 horses, at that rate?

A further such question, interesting in its Old Testament inspiration, requires a geometric progression term, not a sum. The problem is found in the 1849 M'Cord revision of **Willett's Mental and Practical Arithmetic**, and is worded as follows:

3. If the posterity of Noah, which consisted of 6 persons at the flood, increased so as to double their numbers in 20 years, how many inhabiatants were in the world 2 years before the death of Shem, who lived 502 years after the flood?

Solution Noted

Granting some rather remarkable assumptions, solutions are straightforward, as follows.

1. Numbers of cents forming a geometric progression, initial term = 50, common ratio = 2, the young man receives on his 21st birthday a number of cents equal to 50 times the 9th power of 2; that is, 25 600 cents, or $256.00. In all, he receives 50¢ + $1 + $2 + ... + $256, or $511.50.

2. Numbers of cents paid for succeeding horses forming a geometric progression, initial term = 4, common ratio = 4, the gentleman pays for the 15th horse a number of cents equal to the 15th power of 4; that is, $10 737 418.24. In all, 15 horses cost the gentleman a number of cents equal to the sum of the first 15 powers of 4, or $14316557 64 cts.," Ostrander's answer of 1823.

3. The initial 6 persons doubling 5 times each century, 25 times in 500 years, the "posterity of Noah" then numbered 6 times the 25th power of 2, or 100 663 296.

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22. Drama in the Sky

A good story problem could be found by an enterprising nineteenth-century textbook author in almost any daily occurrence. If the chosen happening highlighted the dramatic, of course, then so much the better. An imaginative instance is the Tobias Ostrander quantification and retelling of the saga of the soaring dove. When the hawk "gave the dove a chase," the outcome was all but inevitable, students would have recognized, but the mathematics of simple proportion no doubt proved instructive. Ostrander chose, for variety, to present the question as twelve lines of verse. In this not unattractive form it is found in his **Elements of Numbers** (Canandaigua, N.Y., 1823).

The "chain" being visualized no doubt was the surveyor's chain (or Gunter's chain) of 4 rods, subdivided to 100 links, each (accordingly) 7.92 inches. The dove's headstart, therefore, was 2640 feet, or half a mile.

A harmless dove was soaring high,
To stretch her wings in space--
At length a hawk did her espy,
And give the dove a chase;
Justy forty chains were then between
Those birds as we could view--
When the poor dove flew seventeen,
The hawk just twenty-two;
The hawk pursued with all its strength,
As those who saw did say--
Then tell the chains he flew in length,
Before he caught his prey?

Solution Notes

When the hawk "espied" the dove, the distance between them, we are informed, was 40 chains (0.5 miles). Flying 22 chains to the dove's 17, the hawk reduced the gap by 5 chains for each 22 chains that it flew. Gaining, at this rate, 1 chain for each 4 2/5 chains covered, it wholly closed the gap -- it "caught its prey" -- on flying 40 times 4 2/5, or 176 chains (2.2 miles)

As verification, the hawk flew 176 chains (2.2 miles), the dove flew 17/22 of this, or 136 chains (1.7 miles), and the distance between them thereby was diminished the indicated 40 chains (0.5 miles).

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23. How Many Variations?

David Eugene Smith, mathematics educator and historian, rightly regarded "few algebraic topics more interesting to high school pupils than Permutations and Combinations ... presented in an elementary fashion." Traditional "selections and arrangements" subject matter had been rendered difficult and unpopular by "numerous unnecessary complications" when Smith commented in 1922, however. Whether "taken' as Arithmetic or Algebra, Permutations and Combinations could be a delightful section in earlier works, examination of selected word problems confirms. Most such questions tended to be approached rather "mechanically," one suspects, but related thinking could be interesting, and answers instructive, even surprising.

Nicholas Pike, in his comprehensive (516-page) **New and Complete System of Arithmetick** (Worcester, Mass., 1797), poses this eighteenth-century classic:

1. How many variations may be made of the letters in the word Zaphnathpaaneah?

Stoddard and Henkle, in a **University Algebra** "designed for the use of high schools, academies, and colleges" (New York, 1861), attempt application of combinatoric principles to emerging scientific insight:

2. How many compounds can be formed of 65 elements, each containing two elements, providing each element had an affinity for all the others?

John H. Harney, in his **Algebra** (Louisville, 1840), relates similar thinking to syllable formation, requiring:

3. How many syllables of 3 letters can be formed out of 18 letters, no one of which occurs twice in the same syllable; and no two syllables containing all the same letters?

Arrangement of digits, with restrictions, provides a word problem in Orlando Blanchard, **Theoretical and Practical Arithmetic** (Cazenovia, N.Y., 1854):

4. How many numbers are each expressed by 6 different figures, so arranged as to express the **greatest** number with these figures?

Nineteenth-century story problems often tend to be at their best when they turn to money matters. In his **Improved Arithmetic** (New York, 1828), Daniel Parker applies combinatorics to rustic commerce, as follows:

5. A drover bargained with a gentleman for 12 fat lambs, at $2 a-piece, which he was to select out of 24. Being long in choosing them, the offer was made to the drover, that for **one cent** for every different dozen which might be chosen out of the 24 lambs, he might have the whole; which was readily complied with. What must the drover pay?

Solution Notes

1. Zaphnathpaaneah having 15 letters, including 5 A's, 3 H's, 2 N's, and 2 P's, possible arrangements number factorial 15 divided by the continued product of factorial 5, factorial 3, factorial 2, and factorial 2; that is, 454 053 600.

2. Selecting 2 different elements from 65, order of selection being immaterial, permits 65 x 64/2, or 2080 such binary "compounds."

3. Each selection of 3 different letters from 18 different letters corresponds to a "syllable." Syllables, accordingly, may be formed in 18 x 17 x 16/6, or 816 ways.

4. Six "figures" (digits) to be used (or, equivalently, four **not** to be used) can be selected from 10 digits in factorial 10 divided by the product of factorial 6 and factorial 4; that is, 210 ways. Figures from each selection express the greatest number, of course, when written in decreasing "order of magnitude." This order, in each instance, being unique, digits can be chosen and admissible numbers formed, in 210 times 1, or 210 ways.

5. Different "dozens" could be selected; that is, 12 lambs could be chosen from the 24, in a number of ways equal to the continued product of 13 through 24, divided by factorial 12; that is, in 2 701 456 ways. At one cent per different dozen, the drover pays $12,014.56.

***

Aftermath

The problem-posing skills which "Sums of Yesteryear" so eloquently reflect, have they become an all-but-lost art? On occasion we so fear ... but one delightful question, currently making the rounds, suggests otherwise. Entitled "Mrs. Wilson's Family," it was featured in a "noodlers" challenge selection in a recent **Instructor**, with credit to **Gifted Children Newsletter.** Solution is in small, positive integers, and a trial-and-error approach might suffice, but establishment of variables, and writing and solving of simple equations, provides added challenge.

"Mrs. Wilson's Family" follows:

"Mrs. Wilson had three children: Edgar, Ron, and John. Their combined ages were half of hers. Five years later, during which time Sally was born, Mrs. Wilson's age equalled the total of her kids' ages. Within the following ten-year period another girl was born, Daisy. At the time of Daisy's birth, Edgar was as old as John and Sally together. The combined ages of all the children are now double Mrs. Wilson's age, which is, in fact, only equal to that of Edgar and Ron together. Edgar's age also equals that of the two daughters. If Sally is three years older than Daisy, can you find all of their ages?"

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24. There is a Number

"Digit problems," as a class, still are to be found in school Algebras. Underscoring "place value" in numeration, they provide "word problem" challenge where solution in two or more variables is being "applied." Like other algebraic problem "types," however, digit problems aren't what they once were — which, in this instance, perhaps is as well. In order of difficulty, this quartet of nineteenth-century "digit" challenges derives from (#1, 4) Keller's **Algebra** (Edinburgh, 1869) and (#2, 3) Sargeant's **Elementary Mathematics** (London and Glasgow, 1873).

1. There is a number composed of two digits, whose sum is 7. If we add 27 to the number, the digits will be reversed. Required the number.

2. There is a number of two digits. If the first, or tens' digit, be increased by 5, the number is made three times larger. If the second, or units' digit, be increased by 3, the number is equal to four times the sum of the original digits. What is the number?

3. There is a number of four digits: the sum of the digits is 21; the third digit is twice the first; the sum of the first and last is twice the second; and if 4086 be added to the number, the digits will be inverted. What is the number?

4. There is a number consisting of two digits, which, when divided by the sum of its digits, gives a quotient greater by 2 than the digit in ten's place. But if the digits be reversed, and then divided by a number greater by unity than the sum of the digits, the quotient is greater by 2 than the preceding quotient. Required the number.

Solution Notes

Digit problems developed as textbook applications of simultaneous equations in Elementary Algebra, usual algebraic techniques being expected in their solution. Accordingly, while the base-ten digits necessarily were integers, Diophantine methods ("solution in integers") of Higher Arithmetic or Theory of Numbers normally would not have been employed. Solutions which follow are purely algebraic.

To minimize difficulty in interpretation, for "three times larger," read "three times as large" in = 2; for "inverted" digits, read "reversed" in = 3.

1. Let tens and units digits be t and u. The number is $10t + u$. Reversing digits gives $10u + t$. Then (i) $t + u = 7$; (ii) $(10t + u) + 27 = 10u + t$. Solving, $t = 2$, $u = 5$, and the number is 25.

2. Proceding as in #1, (i) $10(t + 5) + u = 3(10t + u)$; (ii) $10t + (u + 3) = 4(t + u)$. Simplifying, (i) $U = 25 - 10t$; (ii) $u = 2t + 1$. Solving, $t = 2$, $u = 5$, and the number again is 25.

3. Letting thousands', hundreds', tens', and ones' digits be w, x, y, z, the number is $1000w + 100x \pm 10y + z$. Four equations are (i) $w + x + y + z = 21$; (ii) $y = 2w$; (iii) $w + z = 2x$; (iv) $(1000w + 100x + 10y + z) + 4086 = 1000z + 100y + 10x + w$. Substituting for $y = 2w$ and $z = 2x - w$, (i) becomes $2w + 3x = 21$; (iv) becomes $106x - 101w = 227$. Solving, $w = 3$, $x = 5$; and $y = 6$, $z = 7$. The number is 3567.

4. Algebraic solution, not trial and error, being expected, simultaneous quadratics necessarily result. Conditions of the problem give (i) $(10t + u)/(t + u) = t + 2$, which simplifies to $t^2 + (u - 8)t + u = 0$; and (ii) $(10u + t)/(t + u + 1) = (t + 2) + 2$, which simplifies to $t^2 + (u + 4)t + (4 - 6u) = 0$. Subtracting, $u = (12t + 4)/7$. Substituting in (i), $19t^2 - 40t + 4 = 0$, and $t = 2$ or $2/19$. Correspondingly, $u = 4$ or $100/133$. Rejecting fractional values, the number, uniquely, is 24.

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25. On the Road to London

"Elementary" need not be easy, as Kelland, Edinburgh mathematics professor and clergyman, particularly well demonstrates. Kelland's **Algebra** of 1869, a comprehensive treatment of "elements," is highlighted by significant groupings of early, quite challenging, test questions. Deriving from the paper set at St. John's College, Cambridge, in 1815, an unusual "rate" problem, solved in two variables, may be representative. "Elementary" in the sense that its solution calls for rational manipulation, quadratic factorization, and nothing more, the problem (here reproduced) is unlikely to be considered "easy."

"Formation of equations when the conditions are much involved" is necessary in attacking such questions, Kelland points out. Careful reading, clear thinking, unwavering attention, and meticulous care to detail are as needed as algebraic skill, the successful solver soon recognizes. Or, as Kelland observes in discussion, the problems are "admirable in their way, but the skill which they call into play is not altogether of an algebraic character."

As with this Cambridge saga of a London journey, such early test items can be interesting, imaginative, and demanding. Taken at leisure pace, as "problem solving" challenges or as recreational mathematics, they also can be good fun. Attempting such questiosn under competitive examination conditions, Cambridge mathematics students of 1815 may not fully have appreciated this, of course!

The textbook wording of the Cambridge "London road" problem is as follows:

"A and B travelled on the same road, and at the same rate, from Huntingdon to London. At the fiftieth mile stone from London A overtook a drove of geese, which were proceeding at the rate of three miles in two hours; and two hours afterwards he met a stage waggon which was proceeding at the rate of nine miles in four hours. B overtook the same drove of geese at the forty-fifth mile stone, and met the same waggon exactly forty minutes before he came to the thirty-first mile stone. Where was B when A reached London?"

Solution Notes

A and B were travelling at the same rate. Denote this rate by v miles per hour. A remained a constant distance ahead of B. Denote this separation by x miles.

Being "overtaken," the drove of geese was headed in the same direction as A and B, but at a slower rate. A passed the geese at mile stone 50. B passed them at mile stone 45, 5 miles further along. The geese were proceeding at 3 miles in 2 hours, or 5 miles in 3⅓ hours.

At v miles per hour, each traveller covered, in 3⅓ hours, 10v/3 miles. In these 3⅓ hours, we are told, B caught up to A's previous position (mile stone 50) and proceeded 5 miles beyond (to mile stone 45). That is, B covered (x + 5) miles. Equating expressions for distances covered in 3⅓ hours, $10v/3 = x + 5$, and $x = (10v - 15)/3$.

A met the "stage waggon" 2 hours after passing the geese. At v miles per hour, A then was 2v miles past mile stone 50; that is, 50 - 2v miles from London.

Being "met" by A and B, the stage waggon was coming from London. At 9 miles in 4 hours, its speed was 2¼ miles per hour. Adding velocities, waggon and travellers were approaching each other at (v + 9/4) miles per hour.

B met the waggon ⅓ hours, and therefore 2v/3 miles, before mile stone 31; that is, 31 + 2v/3) miles from London.

Subtracting distances from London, the number of miles between the points where A and B met the waggon was (31 + 2v/3) - (50 - 2v), or (8v-57)/3. At 9/4 miles per hour, the waggon covered this distance in (32v - 228)/27 hours. That is, the waggon met B (32v - 228)/27 hours after it met A.

Relative to the two approaching travellers, the waggon advanced at (v + 9/4) miles per hour. It therefore could cover the x miles between the travellers in x/(v + 9/4), or 4x(4v + 9) hours.

Equating expressions for this time interval, 4x1(4v + 9) = (32v - 228)/27, and $x = (32v^2 - 156v - 513)/27$.

Equating this and the earlier expression for x yields a factorable quadratic in v: $(32v^2 - 156v - 513)/27 = (10v - 15)/3$, and $16v^2 - 123v - 189 = 0$. Accordingly, v = 9 or -21/16. Rejecting the negative root as extraneous to the stated problem, the travellers advance at 9 miles per hour.

The constant distance separating A and B (that is, x) is readily given by (10v - 15)/3, or 25 miles (a result which checks in the quadratic expression). When A reached London, therefore, B was on the road, 25 miles (and 2 7/9 hours) from London.

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26. Water in the Hold

Digging a ditch, plowing a field, reaping a harvest, erecting a wall, and filling a cistern were representative tasks in the "work problems" of early Arithmetics and Algebras. With the years, such variants were increasingly encountered, being no doubt more interesting for the solver than the generic "completing a piece of work." One particularly dramatic version of the nineteenth-century "work problem" called for the pumping out of a flooding ship, a veritable "race against time." One uncommonly challenging example is found among test items in Kelland's **Algebra** (Edinburgh, 1869). The question derives from a Cambridge examination of 1817. Though elementary (quadratics do factor!), it calls for meticulous attention to detail. This "story problem," the reader will be pleased to learn, is one with a happy ending. After prodigious effort, and dispite continuing "influx," the vessel indeed is pumped dry.

Wording of Kelland's test item follows:

The hold of a vessel partly full of water (which is uniformly increased by a leak), is furnished with two pumps worked by A and B, of whom A takes three strokes to two of B's; but four of B's throw out as much water as five of A's. Now, B works for the time in which A alone would have emptied the hold; A then pumps out the remainder, and the hold is cleared in 13 hours 20 minutes. Had they worked together, the hold would have been emptied in 3 hours 45 minutes, and A would have pumped out 100 gallons more than he did. Required the quantity of water in the hold at first, and the horary influx at the leak.

Solution Notes

Denote by w the number of gallons of water initially in the vessel's hold, and by x the influx in gallons per hour. Let A pump out y gallons per hour and B pump out z gallons per hour. Denote by h the number of hours in which A alone would have emptied the hold.

Since A takes three strokes to two of B's, and four of B's throw out as much water as five of A's, we have (relation 1): $z = \frac{2}{3}(5/4\ y)$, or $z = 5y/6$.

In h hours, at y gallons per hour, A pumps out hy gallons, an amount corresponding to w gallons initially in the hold plus an influx of hx gallons in the h hours. That is (relation 2): $hy = w + hx$, or $h = w/(y - x)$.

B, in h hours, would remove hz, $h(5y/6)$, or $5(2 + xh)/6$ gallons, leaving $(w + hx)/6$ gallons for A to pump out.

In the remainder of 13 hours 20 minutes; that is, in $(13\frac{1}{3} - h)$ hours, A then pumps out the $(2 + hx)/6$ gallons in the hold plus $(13\frac{1}{3} - h)x$ gallons additional influx. At y gallons per hour, A pumps out $(13\frac{1}{3} - h)y$ gallons. Accordingly (relation 3): $(w + hx)/6 + (13\frac{1}{3} - h)x = (13\frac{1}{3} - h)y$.

Further in $3\frac{3}{4}$ hours, A and B, together, would pump out a number of gallons, $3\frac{3}{4}(6 + z)$, equal to the w gallons initially in the hold, plus a $3\frac{3}{4}x$ gallon influx. That is (relation 4): $3\frac{3}{4}(y = z) = w + 3\frac{3}{4}x$.

A, pumping out $3\frac{3}{4}y$ gallons, would have removed 100 gallons more than the $(w + hx)/6 = (13\frac{1}{3} - h)x$ gallons that he did pump out. That is (relation 5): $3\frac{3}{4}y = (w + hx)/6 + (13\frac{1}{3} - h)x + 100$.

Substituting for $z = 5y/6$ and $h = x/(y - x)$, simplifying, and solving for w, relations 3, 4, and 5 yield expressions in 6 and x. These relations are:

$w = 80(y - x)^2/(7y - 6x)$.

$w = 5(11y - 6x)/8$.

$w = 5(y - x)(240 + 32x - 9y)/(12x - 2y)$.

Equating the first and second expressions for w and simplifying, $51y^2 - 148yx + 92x^2 = 0$, and $y = 2x$ or (extraneously) $y = 46/51x$. Equating the second and third expressions, $25y^2 - 82tx + 92x^2 = 960(6 - x)$. Setting $y = 2x$, $8x^2 = 960x$, and $x = 120$. The "influx," accordingly, is 120 gallons per hour. Taking $y = 2x$, A "pumps out" at 240 gallons per hour. B, correspondingly, pumps at five-sixths that rate, or 200 gallons per hour. Substituting to obtain w, the hold initially contains $5(11y - 6x)/8$, $5(2640 - 720)/8$, or 1200 gallons of water. A alone would pump out the vessel in $w/(y - x)$, $1200/(240 - 120)$, or 10 hours.

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27. A Parcel of Apples

"Records are made to be broken," undeniably, but some of the more startling ones to be encountered in story problems of old mathematics textbooks somehow seem destined to stand the test of time. Two claimants, surely, for world records in apple "devouring" at a single sitting are featured in a textbook review question of 1845. In his **New System of Arithmetic and Mathematics** (4th ed., New York), James H. Porter tells how two men "laid hold of a parcel of apples" — a great many apples — and proceeded to devour them all.

The only recorded aftermath of this fast-paced indulgence was the wonderfully "story problem" revelation that "A had eaten 15 apples more than B" while "A would have been 36 minutes eating B's apples, and B would have required 49 minutes to have eaten A's." The implied algebraic relation is, on simplification, a factorable quadratic. One of the roots is negative and extraneous; the other, very tidily, gives rise to unique, if highly improbable, results.

Two men, A and B, laid hold of a parcel of apples, and after devouring them discovered that A had eaten 15 apples more than B, A would have been 36 minutes eating B's apples, and B would have required 49 minutes to have eaten A's. Required the time they were eating and the number of apples, and how many were devoured by each?

Solution Notes

A and B evidently went on devouring apples as long as apples remained. They therefore were eating for the same length of time.

Denote by x the number of apples devoured by B. A, the evident record-setter, had devoured $x+15$. A eats at a rate of x apples in 36 minutes, or $x/36$ apples per minute. B eats $x+15$ apples in 49 minutes, or $(x+15)/49$ apples per minute.

Accordingly, A eats $x + 15$ apples at $x/36$ apples per minute, and takes a number of minutes equivalent to $x + 15$ divided by $x/36$. Correspondingly, B eats x apples at $(x + 15)/49$ apples per minute, taking a number of minutes equivalent to x divided by $(x + 15)/49$.

Eating times being equal, these expressions for the number of minutes equate. Simplifying, $13x^2 - 1080x - 8100 = 0$, and $x = 90$ or (extraneously) $-90/13$. B, accordingly, devours 90 apples and A devours 105.

The number of apples in the parcel was 195.

Further, A, eating his 105 apples at 2 1/2 apples per minute, and B, eating his 90 apples at 2 1/7 apples per minute, devour their respective quotas in 42 minutes.

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28. A Person's Guineas

Among "curious and entertaining questions relating to numbers," John Bonnycastle, the Woolwich instructor, includes an unusual story problem involving fractional parts. The 1843 London edition of Bonnycastle's **Scholar's Guide to Arithmetic** relates the manner in which a person, visiting a succession of shops to purchase "necessaries," dispensed a supply of gold guineas. Numbers of the coins decreased in an interesting pattern, until finally no guineas remained.

Whether approached arithmetically, as no doubt was intended, or through algebraic expressions and a final linear relation, the problem is elementary, though not without appeal. Classed by Bonnycastle as an "arithmetical recreation," it falls in none of the "problem type" classifications which tended to be stressed at the time.

The traditional 21 shillings value of the gold guinea and its correspondingly imposing purchasing power are worth recalling, but prove irrelevant to solution of the Bonnycastle problem.

A person went out with a certain number of guineas about him, in order to purchase necessaries at different shops; at the first of these he expended half the number he had and half a guinea more; at the second half the remainder and half a guinea more; and so on at a third and fourth shop: at the last of which, having paid for his articles, he found he had laid out all his money: how much had he at first?

Solution Notes

The "certain number of guineas" we denote by n. At the first shop, n/2 + 1/2 guineas were spent, and n - (n/2 + 1/2), or n/2 - 1/2 remained. At the second shop, (1/2) (n/2 - 1/2) + 1/2, or n/4 + 1/4 guineas were spent, and (n/2 - 1/2) - (n/4 + 1/4), or n/4 - 3/4 remained. At the third shop, (1/2) (n/4 - 3/4) + 1/2, or n/8 + 1/8 guineas were spent, and (n/4 - 3/4) - (n/8 + 1/8), or n/8 - 7/8 remained. At the fourth and final shop, (1/2) (n/8 - 7/8) + 1/2, or n/16 + 1/16 guineas were spent, and (n/8 - 7/8) - (n/16 + 1/16), or n/16 - 15/16, remained. At the final shop, however, we are told that no guineas remained. Accordingly, equating, n/16 - 15/16 = 0, and n = 15. Initially the person had "about him" fifteen guineas. Expressions for numbers of coins spent and remaining a 11 denote integers. Thus, at the first shop, 7½ ÷ ½, or 8 coins were spent, and 7 remained; at the second shop, 3½ + ½, or 4 coins were spent, and 3 remained; at the third shop, 1½ + ½, or 2 coins were spent, and 1 remained; and at the final shop, ½ + ½, or 1 coin was spent, and none remained.

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29. What Number is That?

"I am thinking of a number," we might choose to begin.

Early Arithmetics adopted a diversity of wordings when they posed questions from this interesting, neglected category — story problems concerning attributes, operations, and relations of numbers themselves.

Francis Walkingame, in his successful and long-lived **Tutor's Assistant** (London, 1846), incorprates (as review work) a score or more such "number questiions." We offer a perhaps revealing selection.

The problems seem simple, and no doubt are; so no "solution notes," only Walkingame's answers, are appended. Should doubts arise, our reader, like Walkingame's early tutor, can "work backwards," confirming for himself in the process the author's likely intent.

A perfect score should attest to careful reading, incisive thinking, and corret (if rudimentary) work with figures.

1. What number is that, from which if the square of 14 be deducted, and to the remainder the square of 12 be added, the sum will be 250?
2. What number added to the cube of 21, will make the sum equal to 11 times 147?
3. What number taken from the square of 54, will leave 19 times 46?
4. What fraction is that which being multiplied by 2/3, will produce 1/4?
5. The less of two numbers is 187, their difference 34, the square of the product is required?
6. What is the difference between 14676, and the fourth of itself?
7. What number deducted from the 26th part of 2262, will leave the 87th part of it?

The author

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8. What number is that which makes 9 to be the 2/3 of it?

9. There are two numbers, the one 48, the other twice as much; what is the difference btween their sum and difference?

10. There are three numbers, 17, 19, and 48; I demand the difference between the sum of the squares of the first and last, and the cube of the middlemost?

11. There are two numbers, the one 63, and the other 1/2 as much; I demand the product of their squares, and the difference of their product and sum.

12. If 48 taken from 120 leaves 72, and 72 taken from 91 leaves 19, and 7 taken from thence leaves 12; what number is that out of which when you have taken 48, 72, 19, and 7, 12 is left?

Answers and Observations

(1) 302, the number; (2) 7350, the number; (3) 2042, the number; (4) 3/8; (5) 1707920929; (6) 11007, difference; (7) 61, the number; (8) 13½, the number; (9) 96; the difference; (10) 4266, the difference; (11) product of their squares, 3938240.25; difference, 1890; (12) 158, the number.

Lest one conclude that all Walkingame number reckoning was this rudimentary, the author, in the same slim volume, does require multiplication of 909290929090909099 by 132118812, in four lines of products; also, without in-having introduced logarithms, the square root of 6 2/5, the cube root of 8 5/7, the sixth root of 3.1416, and (perhaps most remarkably) the 40th root of 1.04.

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30. Bacchus and Silenus Empty the Cask

John Herbert Sangster, nineteenth-century physician, provided mathematics textbooks for a generation of secondary schoolers of several Canadian provinces, and he created or adapted his "story problems" with a particular flair. Sangster's **Elements of Algebra** (Montreal, 1865) offers an interesting illustration. Two of the more colourful, if less upstanding, personages of Greek and Roman mythology feature in a Sangster "word-problem" with what Sangster may have meant as moral overtones. Bacchus and Silenus, each at his own pace, engage in the kind of shared task typical of "work" problems, the emptying of a full cask (of wine or ale, presumably) by the consumption of its contents, at constant, to-be-determined rates.

Solution, typically, is algebraic, with the relationships non-linear. Emphasis is on reciprocals, and higher degree terms, to cubic, involve products of variables. (When the mythological duo imbibe together, drinking time actually is half the harmonic mean of individual times.) Numbers, however, are simple throughout, and elementary factoring (or the Victorian "completion of the square") yields a ready solution. However, one slip along the way and, not suprinsingly, computation can become remarkably involved.

The temperance lesson implicit in Sangster's story problem may or may not have been effective. The associated workout in higher-degree simultaneous equations is, however, a relatively thorough one, though to simple numerical ends.

Sangster's wording of 1865:

"Bacchus caught Silenus asleep by the side of a full cask, and seized the opportunity of drinking, which he continued for 2/3 of the time that Silenus would have taken to empty the whole cask. Silenus then awoke and drank what Bacchus had left. Had they both drank together, it would have been emptied two hours sooner, and Bacchus would have drunk only half what he left Silenus. How long would it have taken each to empty the cask separately?"

Solution Notes

Denote by h and k the number of hours in which the cask would have been emptied by Silenus and Bacchus, respectively. Each hour, Silenus would drink 1/h of the cask and Bacchus 1/k of the cask.

In two-thirds the time that Silenus would take to empty the cask; that is, in 2h/3 hours, Bacchus would consume (2h/3) (1/k), or (3/k - 2h)/3k of the cask. His having done so would leave for Silenus 1 - (2h/3k), or (3k - 2h)/3k of the cask.

Since Silenus would empty the cask in h hours, he would drink this (3k - 2h)/k of the cask in h(3k - 2h)/3k hours.

Accordingly, emptying the cask would take, in all, 2h/3 + h(3k - 2h)/3k, or h(5k - 2h)/3k hours.

Should Silenus and Bacchus drink together, emptying the cask in, say, n hours, then, for each hour, 1/h + 1/k = 1/n; giving n = hk/(h + k) hours, one-half the harmonic mean.

This drinking time, we are told, is two hours less than the time actually taken. Equating numbers of hours, hk/(h + k) = h(5k - 2h)/3k - 2; or $3h^2k + 2hk^2 = 2h^3 + 6hk - 6k^2$.

Further, in the hk/(h + k) hours, at 1/k cask per hour, Bacchus would have downed h/(h + k) of the cask. This amount represents, we are told, one-half of the (3k - 2h)/3k cask left for Silenus. Equating, h/(h + k) = (3k - 2h)/6, or 3k2 - 5hk - $2h^2 = 0$. Solving, k = 2h or (extraneously to the problem) k = - h/3. Substituting in the previous relation, $12h^3 = 36h^2$, giving h = 3 or (extraneously) h = 0. Accordingly, Silenus would have emptied the cask in three hours, Bacchus in six.

"Checking" proves straightforward. In two hours Bacchus drank one-third of the cask, leaving two-thirds for Silenus. Silenus then consumed that two-thirds cask in two hours. Imbibing together, they would have downed the contents in two hours, two hours less than the four hours actually required. Further, Bacchus, drinking one-third, actually did drink one-half the two-thirds he left for Silenus, the second stipulation.

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31. Water In The Well

Examination papers in Elementary Algebra which date from the period when Algebra was a university subject, commonly put a premium on originality, featuring questions and solution techniques that were anything but routine. Such questions need not have been unduly difficult, of course, though frequently they were, but they did call for both an unusual depth of understanding and meticulous manipulation of symbols.

One such question of unusual interest, a "measures" problem, was included in James Wood, D.D., **Elements of Algebra**, the London edition of 1861. Wood had been Master of St. John's College, Cambridge, and the question, of perhaps average difficulty, had appeared on a paper of 1857. The wording is straightforward, and much of the interest is in the manner in which reciprocals provide a useful approach. Elementary, but probably nonroutine, manipulation of four such variables gives insight into the kind of originality being expected of the Elementary Algebra student at this time.

The St. John's College "measures" problem, as given by Wood:

"Alfred, Edward, and Herbert come each with his pail to a well; when a question arises about the quantity of water in the well: but none of them knowing how much his pail will hold they cannot settle the dispute. Luckily Mary comes up with a pint measure, by aid of which they discover that Alfred's pail holds half a gallon more than Edward's, and a gallon more than Herbert's: but before the precise content of any pail is found out an accident happens, and Mary's measure is broken. They are now however in a position to ascertain the quantity of water in the well: for they find that it fills each pail an exact number of times; and that the number of times it fills Edward's is greater by eight than the number of times it fills Alfred's and less by forty than the number of times it fills Herbert's. How much water was there in the well?"

Solution Notes

Denote by x, y, z, respectively, the capacities, in gallons, of Alfred's, Edward's, and Herbert's pails. Then: x =y + 1/2, and x = z + 1.

Denote by w the number of gallons of water in the well. This volume will fill the pails the integral numbers of times given by, respectively, w/x, w/y, and w/z. Further given conditions relate these integral values as follows: w/y = w/x + 8, and w/y = w/z - 40.

We have four independent, consistent equations in four variables.

Working initially with the reciprocals of the variables will be successful, and possibly instructive. The first two relations give $1/y = 2/(2x - 1)$, and $1/z = 1/(x - 1)$. The third and fourth relations may be rewritten $w(1/y - 1/x) = 8$, and $w(1/z - 1/y) = 40$. From division, $1/z - 1/y = 5(1/y - 1/x)$. Substituting for $1/x$, $1/y$, and $1/z$, and simplifying: $x(2x - 1) = 12x(x - 1) - 5(x - 1)(2x - 1)$. Quadratic terms "cancelling", x, uniquely, is 5/4. On further substitution, $y = 3/4$, $z = 1/4$. From the third relation, $w(4/3 - 4/5) = 8$; from the fourth, $w(4 - 4/3) = 40$: in either instance, $w = 15$.

The boys' pails hold 5 quarts, 3 quarts, and 1 quart, respectively. The water in the well amounts to 15 gallons.

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32. A Scrutiny being Demanded

"Plumper!" There's a good, old-fashioned term and concept for the student of the political process. "A vote given to one candidate only, when two or more are to be elected," according to our dictionary. Or, more broadly, where two or more votes can be cast. Thus, "giving a plumper," the elector opts to cast one vote for one candidate, when he might have marked his ballot for each of several. A certain added advantage for the candidate is seen to result.

"Plumpers" feature in an involved but elementary "election" question included by Colenso (**Elements of Algebra**, London, 1873). From an early Cambridge "equation paper," it solves in six variables, but equations are linear and variables necessarily integers. An election to the House of Commons is under consideration, with a recount. Knowing the exact meaning of "plumper" is essential to solution.

The Cambridge question:

"At an election of one member to Parliament, one-third of the electors gave plumpers for C, and those given for A and B were 4/21 of the whole number of votes given. Of those electors who gave single votes to C, twice as many voted for B as for A: and B stood at the head of the poll with a majority of 110 over C. A scrutiny being demanded, it appeared that those who had split their votes between A and B had no legal right to vote, and C is now returned with a majority of 200 over A. Find the final state of the poll, it being observed that A has now as many single votes as plumpers."

Solution Notes

On the original tally of votes, let numbers of plumpers for A, B, and C be a, b, and c, respectively; and of "split votes" for A and B, B and C, and C and A be m, n, and r, respectively. A number of electors equal to $a + b + c + m + n + r$ gives votes numbering $a + b + c + 2m + 2n + 2r$; that is, $a + m + r$ votes for A, $b + m + n$ votes for B, and $c + n + r$ votes for C. After the "scrutiny," however, m becomes zero, reducing numbers of votes for A and B to $a + r$ and $b + n$, respectively.

Six linear equations in six variables readily follow: $(1/3)(a + b + c + m + n + r) = c$, or $2c = a + b + m + n + r$ (I); $(4/21)(a + b + c + 2m + 2n + 2r) = a + b$, or $17a + 17b = 4c + 8m + 8n + 8r$ (II); $n = 2r$ (III); $b + m + n = (c + n + r) + 110$, or $b + m = 110 + a + c$ (IV); $c + n + r = (a + r) + 200$, or $c + n = a + 200$ (V); and $a = r$ (VI). Equations IV, VI, and III yield useful expressions for m, r, and n, namely: $m = 110 + a + c - b$, $r = a$, and $n = 2r = 2a$. Substitution of these expressions permits solution of the three remaining equations (I, II, and V) for numbers of plumpers, a, b, and c. On substitution, I, II, and V become $c = 5a + 110$, $25b = 15a + 12c$, and $c = 200 - a$. Equating expressions for c, $5a + 110 = 200 - a$, and $c = 15$. Accordingly, $a = 185$. Substituting for a and c in $25b = 15a + 12c$, $b = 133$. In all, 333 plumpers were cast. Values of m, n, and r follow: $m = 110 + a + c - b$, or 177; $n = 2a$, or 30; and $r = a = 15$. On the initial count, therefore, votes for A were $a + m + r$, or 207; for B, $b + m + n$, or 340; and for C, $c + n + r$, or 230. The scrutiny disallowed the split votes of $m = 177$ electors, reducing by 177 the totals for A and B.

Accordingly, the "final state of the poll" was: for A, $a + r$, or 30 votes; for B, $b + n$, or 163; and for C, $c + n + r$, or 230, results which satisfy the original problem conditions and the six independent equations.

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sums of yesteryear

By Don Allen

Recalling the old and the unusual in traditional school mathematics, "Sums of Yesteryear" poses and sets out to solve early "story problems" from diverse original sources.

33. From Collingwood to Goderich

Victorian England provides the setting for virtually all the most interesting "story problems." The preparation of such questions had become a highly-developed skill in rather restricted academic circles, and many review problems in college and school Algebras, in Britain and overseas, derived or were adapted from certain English university examinations. Hence the English settings. Some notable exceptions are due to John Herbert Sangster, the physician-author, who "set" the more original of his problems in pre-Confederation Canada. A question concerning the stagecoach from Collingwood to Goderich, in Upper Canada (later Ontario) appeared in Sangster's Elements of Algebra (Montreal, 1865), and is perhaps typical. A certain inelegance may be sensed in the interrelationship of three necessary variables, but solution follows with little difficulty. The setting in itself assures a certain antiquarian charm.

"A stage set out from Collingwood to Goderich with a certain number of passengers, 4 more being outside than inside. The fare of seven outside passengers is half-a-dollar less than that of 4 inside passengers, and the whole fare received amounted to $45. At the end of half the journey it took up three more outside and one more inside passenger, in consequence of which the whole fare received was 1 2/15 times what it was before. What was the number of passengers and the fare of each?"

Sangster's dollar fares are significant. Canada had been on decimal currency (rather than pounds, shillings, and pence) since 1858. "Dollars and cents" could give Sangster's works an edge over Canadian printings of English texts.

Assume, as would seem reasonable, that a passenger travelling half the distance would pay half the appropriate Collingwood-Goderich fare.

Solution Notes

Denote by n the initial number of inside passengers. The corresponding number of outside passengers will be n + 4.

Denote numbers of dollars in "outside" and "inside" Collingwood-Goderich fares by h and k, respectively. By assumption, those "taken up" for the second half of the journey pay numbers of dollars given by h/2 and k/2.

Then, seven outside passengers would pay 7h dollars, four inside passengers would pay 4k dollars, and 7h = 4k - 1/2, or 14h = 8k - 1.

Further, n + 4 outside passengers, at h dollars, would pay h(n + 4) dollars; n inside passengers, at k dollars, would pay kn dollars. Accordingly, h(n + 4) + kn = 45, from which n = (45 - 4h)/(h + k).

Three "outside" half-fares plus one "inside" half-fare increase the total to "1 2/15 times what it was before." The "whole fare received" having been $45, the increment is (2/15) ($45), or $6. Equating, 3h/2 + k/2 = 6, or k = 12 - 3h.

Substituting for k in the first relation, 14h = 8 (12 - 3h) - 1, and 38h = 95. Hence, h = 2 1/2, and the outside fare is $2.50. Accordingly, k = 12 - 3(2 1/2), k = 4 1/2, and the inside fare is $4.50. Further substituting, n = (45 - 4h)/(h + k) = 35/7 = 5. Five inside passengers, and hence nine outside passengers, set out from Collingwood for Goderich.

Responding to Sangster's question: there were, in all, 18 passengers; travelling the full distance were nine outside passengers at $2.50 and five inside passengers at $4.50; travelling the second half of the journey were three additional outside passengers, paying $1.25, and one additional inside passenger, paying $2.25.

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34. Twelve "Examination Exercises"

Problem solving has been identified as a priority for school mathematics in the 1980s, but the compilation of worthwhile, non-routine problems has been a concern of textbook authors for a long time. In his **New Arithmetic** (Toronto, 1885), a remarkable anthology, Seymour Eaton assembled and edited contributions of "three hundred prominent educators." The following selection of shorter "examination questions" from this source suggests the diversity of non-routine arithmetic problems being circulated in nineteenth-century Canada.

1. What number must be added to 20009 to make it exactly divisible by 201?
2. Find the smallest multiplier that will make 3428 a perfect square.
3. What number multiplied by 3/11 of itself will give 1188?
4. Find the sum of all the prime numbers between 125 and 160.
5. Find the sum of the five largest numbers that can be expressed by the figures 2, 1, 0, 3 and 7.
6. The sum of two numbers is 100020002; their difference is 90009001. Divide twice the larger by ½.
7. How many miles will a boy walk to plow 6 acres, turning a furrow of 9 inches?
8. A substance is weighed from both arms of a false balance, and its apparent weights are 18 pounds and 8 pounds. Find its true weight.
9. A certain sum amounts to $2070 in 6 months, and to $2420 in 3 years, simple interest. What is the rate per cent?
10. In 1880 February had 5 Sundays. When will this occur again?
11. A regiment marching 4 miles an hour take 128 steps in a minute. What is the length of each step?
12. How far must a person proceed on a stage which travels 4½ miles an hour in order that he may walk back at the rate of 3 miles an hour and be gone just 7½ hours?

The Eaton compilation being a school Arithmetic, arithmetic methods would, in general, be expected.

Solution Notes

1. Dividing, 20009 is 99 x 201, with 110 remainder. Adding 201 - 110, or 91, accordingly, gives 20100, or 100 x 201, the next multiple.
2. Prime factorization gives 3428 = 2 x 2 x 857. multiplication by an additional factor of 857 therefore will be necessary to yield a perfect square.
3. Correspondingly, 1188 divided by 3/11, or 4356, will equal the number times itself. The number is the square root of 4356, or 66.
4. "Sifting" or running standard divisibility tests, the seven primes in the stipulated range are 127, 131, 137, 139, 149, 151, and 157. Their sum is 991.
5. Using the figures as digits and reserving larger digits for the higher place values: 73210 + 73201 + 73120 + 73102 + 73021 = 365654.
6. "Twice the larger" can be found directly by adding sum and difference: 190029003; dividing by ½ (that is, doubling) then gives 380058006.
7. The problem can be worked in yards and square yards. One acre being 43 560 square feet, or 4840 square yards, 6 acres is 29 040 square yards. The furrow width is 9 inches, or ¼ yard. To equate to 6 acres, accordingly, total furrow length must be 29 040 square yards divided by ¼ yard, or 116 160 yards. Converting, 66 miles exactly.
8. False balances are assumed to err in constant proportion. The true weight is to 18 pounds as 8 pounds is to the true weight, giving the weight as the mean proportional, 12 pounds.
9. The $350 difference between the three-year and six-month amounts represents 2½ years' simple interest. Annual interest, therefore, is $140. The original sum is $2070 - ½ ($140), or $2000. Annual interest of $140 on $2000 reflects a rate of seven per cent.
10. The calendar "cycle" normally would be seven fours, 28 years. However, 1900, being a "century not divisible by four," was **not** a leap year. We leave the question open to discussion.

11. Four miles per hour are 21 120 feet per hour, or 352 feet per minute. Accordingly, 128 steps cover 352 feet; each step is 352/128 feet, or 2 feet 9 inches.

12. The stage, at 4½ miles per hour, takes 2/9 hour to go a mile. Walking, at 3 miles per hour, takes 1/3 hour to go the same distance. Accordingly, for each mile the person must allow 2/9 + 1/3, or 5/9 hour. The number of miles the person must proceed is 7½ divided by 5/9; that is, 13½ miles.

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35. Four Ontario Classics

John Herbert Sangster, the Confederation-era Canadian textbook author-physician, made use of contemporary "Upper Canada" settings for a number of story problems in his Elements of Algebra (Montreal, 1865). Familiar place names and nostalgic circumstances enhance the interest and charm of such problems. Number work is a bit awkward, however, and the solver cannot depend upon the overly simple answers commonly encountered in story problems of this period. Four problems are illustrative. The first three involve only "simple equations" -- that is, equations in one or more variables, all (on reduction) of the first degree. The fourth problem is quadratic, and Sangster would seem to have overreached himself. Either he has fumbled in setting his "travel on foot" word problem, or brisk jogging came to Central Ontario considerably earlier than sportswriters record.

The four problems with Confederation-era "Upper Canada" settings:

1. **The Canal and the River.** The Rideau Canal is six miles less than four times as long as the Niagara River, and their combined length doubled and decreased by 100 miles, exceeds the length of the Great Western Railway by one mile. The G.W.R. being 229 miles long, required the length of the Rideau Canal, and also that of the Niagara River.

2. **From Toronto to Brampton.** A person starts to walk from Toronto to Brampton at the rate of 3½ miles per hour; precisely 28½ minutes afterwards another person starts from Brampton to walk to Toronto at the rate of 4 miles per hour, and they meet one another exactly halfway between the two places. Required the distance from Toronto to Brampton.

3. **The Lake Ontario Steamer.** A steamer makes the down trip from the head of Lake Ontario to Montreal in 28 hours, the current being in its favor. When returning it is found that in ascending the St. Lawrence (three-sevenths of the entire trip) the rate of sailing is 5 miles per hour less than the average rate on its downward journey, but upon entering the lake it is enabled to increase its speed 2 miles per hour, and again reaches Hamilton, at the head of the Lake, in 19/21 of the time it would have required had the rate been uniformly the same as when ascending the river. Required the distance between Montreal and Hamilton, and the rates of sailing.

4. **Two Persons On Foot.** Two persons set out at the same time to travel on foot, A from Toronto to Cobourg, and B from Cobourg to Toronto. When they meet it is found that A has travelled 15 miles more than B, and that A will reach Cobourg in 2 hours, and B, Toronto in 4½ hours. Find the distance between Toronto and Cobourg and the rate of travelling of each.

Solution Notes

1. Represent the lengths of the river and the canal by r and 4r - 6 miles. Then, 2(5r - 6) - 100 = 229 + 1, and 10r = 342. The Niagara River is 34 1/5 miles long. The Rideau Canal is 130 4/5 miles long.

2. Suppose the second person walks for t minutes, the first for t + 28½ minutes. The meet half-way, so distances they walk are equal. These distances are products of rates (in miles per hour) and times (in hours). At 3½ miles per hour, the first person covers (3½) (t + 28½)/60 miles. At 4 miles per hour, the second person covers 4t/60 miles. Equating and multiplying by 60, (3½) (t + 28½) = 4t, and t = 199½ minutes. Each covers 13 3/10 miles. The distance from Toronto to Brampton is 26 3/5 miles.

3. Take the distance between Hamilton and Montreal to be d miles. Represent the average rate of the steamer from Hamilton to Montreal by r miles per hour. The down trip taking 28 hours, and "time" being distance divided by rate, 28 = d/r, and d = 28r. For the up trip, 3d/7 miles is travelled at r - 5 miles per hour, then the remaining 4d/7 miles at r - 3 miles per hour. Total time, accordingly, is (3d/7)/(r - 5) + (4d/7)/(r-3) hours. This total time equates to 19/21 of the time for d miles at r - 5 miles per hour; that is, (19/21)(d/)r - 5)). Equating and simplifying, r = 15 miles per hour; d = 28r = 420 miles. Accordingly, the distance between Montreal and Hamilton is 420 miles; the down trip is at r, or 15 miles per hour; and the up trip, on the river, is at r - 5, or 10 miles per hour, and on the lake is at r - 3, or 12 miles an hour. (The up trip, accordingly, takes 10 hours longer than the down trip, or 38 hours.)

4. Suppose A "walks" at m miles per hour, B at n miles per hour. When they meet, B has covered d miles A has covered d + 15 miles. Walking times being equal, (d + 15)/m = d/n. Further, A at m miles per hour, will cover d remaining miles in 2 hours; so d/m = 2. Correspondingly, B at n miles per hour, will cover d + 15 remaining miles in 4½ hours; so (d + 15)/n = 4½. Setting d = 2m, n = (2/9) (2m + 15), and simplifying, m^2 - 12m - 45 = 0, and m = 15, or (extraneously) - 3. Substituting, n = 10, d = 30. A, accordingly, must be moving at 15 miles per hour. From Toronto to Cobourg, 2d + 15 miles, is 75 miles.

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AFTERWORD

> If 12 oxen eat up $3\frac{1}{2}$ acres of grass in 4 weeks, and 21 oxen eat up 10 acres in 9 weeks, how many oxen will eat up 24 acres in 18 weeks; the grass being at first equal on every acre, and growing uniformly?

There's good fun in these old problems--or, at least, as we distract colleagues with such a challenge at the Arctic College photocopier, we find we're not alone in this belief. The unusual "rate problem" reproduced above (enlarged from the actual 1837 text-book page) derives from Emerson's Arithmetic, and was on the back of our business card for some years. Like the examination questions which follow, it might seem a bit difficult for the general reader, but should challenge and amuse those likely to assemble in Iqaluit on this occasion. Actually, as a teacher would recognize, there's much to be learned from even the more routine story problems which we shared in The Teacher in our Nova Scotia days. As one instance, the fourth section of our "Walkingame Sampler" (Column 4) caught our attention as we proceeded with this ten-years-after paste-up. An attractive, alternate solution, without the algebra, works backwards: if the 20 remaining sheep, plus half a sheep more, double to the previous number of sheep, then there were 41 sheep. Prior to that, if 41 sheep, plus half a sheep more, double to the preceding number, then there were 83 sheep. Those 83 sheep, plus half a sheep more, doubling to the required original number, there initially were 167 sheep, which is Walkingame's published result. Now, these old problems may "recycle" fairly readily: replace "halve" by "decimate" in the story line, substitute "caribou" for "sheep," introduce predatory wolves, and the story becomes contemporary and assumes "northern relevance."

These days of "accelerated math" programs, high school Calculus, and such, know little of the mental workout inherent in a really good story problem in Arithmetic or Elementary Algebra. An intriguing source is such a compendium as Colenso's Algebra which incorporates lengthy groupings of "examination questions." The following are attributed to Cambridge University entrance papers of the mid-nineteenth century.

> Three boats, A, B, and C, start in a race, B being 20 yards behind A, and C the same behind B. A and C set off at an uniform rate, C making a yard less *per* stroke than A. But B took 7 strokes to 6 of A or C, and increased its speed besides by 3 in. every stroke; so that, when A had taken 42 strokes, B, which had lost 16 yards by steering, was only a yard behind A. At this point B's speed decreased twice as fast as it had increased before; while C, quickening its strokes at the same instant in the ratio of 6 : 7, and gaining each stroke as much speed as B lost, at the end of 28 strokes overtook B, which had lost 11 yds more by steering. Compare the velocities with which they started.

In the first and least considerable irruption of the Thames into the Tunnel, the water rose in the vertical shaft 8 times as fast as in the horizontal levels in the second. If the levels at the second influx had been 110 feet longer, the velocities of the water ascending in them in the first and second irruptions, and when thus increased would have formed an A.P., the common difference being $\frac{1}{9}$ of the difference of the velocities with which the water rose in the shaft in the two irruptions; and, if the levels had been of the same length on both occasions, the first time of filling would have been half as long again.

The tunnel consisted of two equal levels, terminated by a vertical shaft of twice the breadth of either. The sections of the shaft and levels are supposed to be squares; and the height of the shaft above the upper surface of the levels to be double of its breadth. Given the first time of filling to be 10′ less than the second, find the duration of the latter.

A regiment, in which there are between 10 and 100 officers and twice as many serjeants, in clearing the streets during a revolution, loses 2 officers; and, after storming a barricade in which three more fall, is obliged to retreat, taking in a volunteer as officer, but, in so doing, loses other three. While clearing the streets, the liability of an officer to fall is half that of a serjeant or private; but at the barricade as 4 : 3, and in the retreat as 3 : 4. Also, on their leaving the barracks, the number, whose two left-hand digits express the number of serjeants and its other digits that of officers, exceeds by 20 ten times the number of privates; but, on their return, (having parted with the volunteer,) it exceeds it only by 12, the number of officers being still above 10. Find the state of the regiment at first.

The income of a schoolmaster arises partly from ten pupils residing in his house; and partly from an endowment, which produces a certain number of quarters of wheat each year. When wheat sells for 60*s* the expenditure of his family (£249) is less than his savings by a number, which when divided by twice the number of his pupils expresses the proportion which the clear gain bears to the whole charge for each pupil. In the following year wheat falls to 55*s*, and a tax of 8*d* in the pound is laid upon income, payable upon the net income of the preceding year; but the cost of living for his pupils being diminished, (so that, in fact, the amount of income-tax he has to pay, with 10*s* added, would just support one pupil,) he finds that his savings are greater than in the year previous by a sum equal to the difference of his net income in the two years, which is $\frac{1}{18}$th of the expenditure of his family in the second year, besides allowing for an outlay of £15 in repairs. The net income from pupils in the first year being £330, find that from the endowment in the same year and the ratio of the costs of living in the two years.

A and *B* set out together, and walk from Keswick over Helvellyn to Ambleside. *B* arrives at the foot of Helvellyn, which is 5 miles from Keswick, in an hour, and then slackens his pace so that he just reaches the top at the end of the second hour. Here he sits down to rest, until *A* passes him and gets as much before him (*B*) as he was behind when *B* first sat down. *B* then starts at an increased pace, and passes *A* at the end of the third hour from the time of starting. *B* walks on at the same pace for another hour, and then waits for *A* to come up, whose distance behind him was $\frac{3}{5}$ of what it was at the end of the first hour. When *A* overtakes him, he starts again and walks at the same pace until he reaches Ambleside; the time of this last stage being to that of his first rest in the ratio of 20 : 7. On his arrival, he fires a pistol for the information of *A* who, having hitherto kept up a uniform pace without stopping, now diminishes it in the ratio of 5 : 7, and reaches Ambleside 10 minutes after *B*. What is the distance from Keswick to Ambleside by this route?

A man, who is not aware that his watch gains uniformly, engages to ride from Cambridge to London in 9 hours, and sets his watch by St. Mary's at starting. Upon looking at it after having gone halfway, he supposes it necessary to increase his pace in the ratio of 4 : 3, and consequently reaches London 15′ within the time. But if the watch had lost at the same rate, and he had looked at it at the end of the 14th mile, and then regulated his pace accordingly, he would have been in London too late by 7′. Find the distance from Cambridge to London.

A stable-keeper bought two horses for £50, and sold them, one for double, and the other for half, of what he gave for it. The former had produced for its hire only half of what it cost him, and cost in keep as much per cent on its price as the hire of the other produced on its price, the latter being kept for $\frac{4}{5}$ as many guineas as it sold for pounds. The keep of the two cost £33, and he made by them, upon the whole transaction, nine times his profit on the sale. What did each cost?

Enjoy!

Iqaluit, 30 March 1994.

MENTAL ARITHMETIC.

The following *table* will be found useful.

Multiplication and Division Table.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
3	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60
4	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80
5	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
6	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120
7	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133	140
8	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152	160
9	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144	153	162	171	180
10	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
11	11	22	33	44	55	66	77	88	99	110	121	132	143	154	165	176	187	198	209	220
12	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240
13	13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247	260
14	14	28	42	56	70	84	98	112	126	140	154	168	182	196	210	224	238	252	266	280
15	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285	300
16	16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	272	288	304	320
17	17	34	51	68	85	102	119	136	153	170	187	204	221	238	255	272	289	306	323	340
18	18	36	54	72	90	108	126	144	162	180	198	216	234	252	270	288	306	324	342	360
19	19	38	57	76	95	114	133	152	171	190	209	228	247	266	285	304	323	342	361	380
20	20	40	60	80	100	120	140	160	180	200	220	240	260	280	300	320	340	360	380	400
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Smith and McMurchy, Elementary Arithmetic, Toronto, 1869. OISE Collection.